Minimal Presentation Sizes of Numerical Semigroups

Elmacioglu Ceyhun, Park-Kaufmann Hannah, Okandan Melin
Lafayette College, Bard College, Bilkent University

Abstract

It is a well-known fact that if m is the multiplicity of a numerical semigroup S, then the size of the minimal presentation is at most $\binom{m}{2}$. We introduce a combinatorial approach involving posets to determine the attainable minimal presentation sizes given a fixed m, which has been a long-standing open problem.

Our Terminology

- A **numerical semigroup** is a cofinite subset of positive integers closed under addition.
  $$S = \langle 6, 9, 20\rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$$
- Each numerical semigroup has a minimal set of generators. The **multiplicity** is the smallest element in the set of generators.
  $$S = \langle 6, 9, 20\rangle \implies m = 6$$
- The **embedding dimension** is the number of elements in the minimal set of generators.
  $$S = \langle 6, 9, 20\rangle \implies e = 3$$
- A **trade** is two distinct ways of writing the same number using the minimal set of generators.
  $$S = \langle 6, 9, 20\rangle \implies 18 = 3 \cdot 6 = 2 \cdot 9$$
- The **minimal presentation size** of a numerical semigroup is the smallest number of trades in terms of which all other trades can be written as a combination.

Acknowledgement

This research was conducted under the guidance of Professors Christopher O'Neill and Tara Gomes during the Polymath REU 2021.

References

1. Gomes, T., O'Neill, C., Davila, E. T. (2020). Numerical semigroups, polyhedra, and posets III: minimal presentations and face dimension. arXiv preprint arXiv:2009.05921.
2. Rosales, J.C. "Numerical Semigroups with Apéry Sets of Unique Expression." Journal of Algebra, vol. 226, no. 1, 2000, pp. 479–487., https://doi.org/10.1006/jabr.1999.8202.

Kunz Poset

Given a numerical semigroup S with multiplicity m, the **Kunz poset** is the partially ordered set with ground set $\mathbb{Z}_m$ obtained by the equivalence class in $\mathbb{Z}_m$.

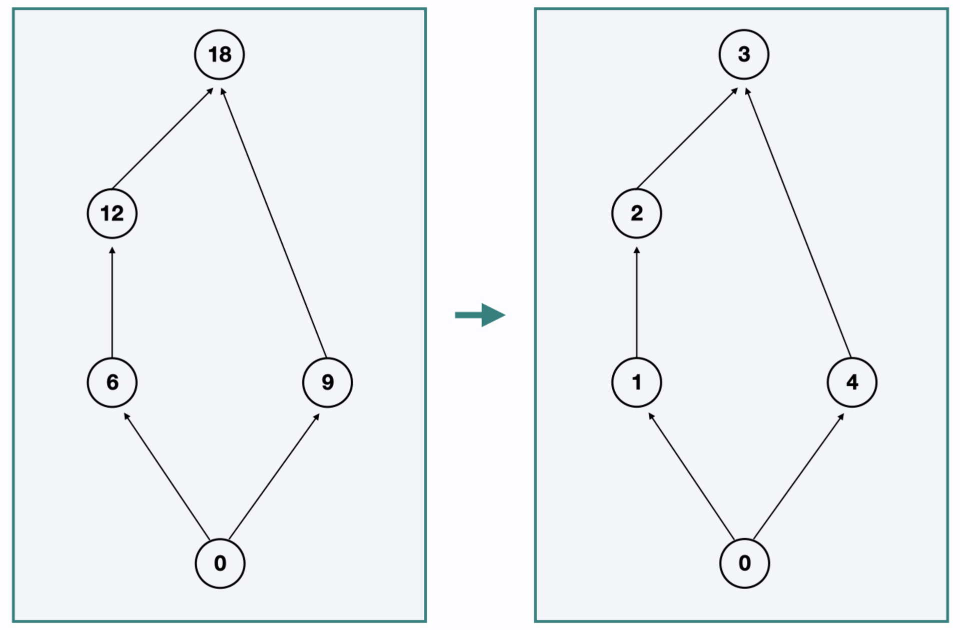

Figure 1: Poset of $\mathbb{Z}_m$ (left) and Kunz Poset of $S = \langle 5, 6, 9\rangle$ (right)

Main Theorem (EKO)

The minimal presentation size of a numerical semigroup S is equal to the sum of the number of outer Betti elements and relations of its Kunz poset P.

Outer Betti Elements

An **outer Betti element** of P is a factorization missing in the Kunz poset that is connected by a single step to an element of the Kunz poset.

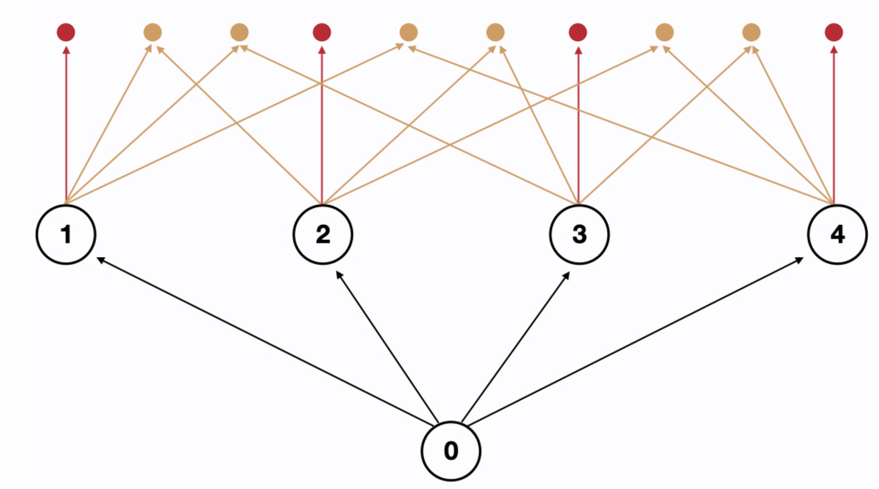

Figure 2: The $10 = \binom{4}{2} + 4 = \binom{5}{2}$ outer Betti elements in Kunz Poset of $S = \langle m, m+1, m+2, m+3, m+4\rangle$

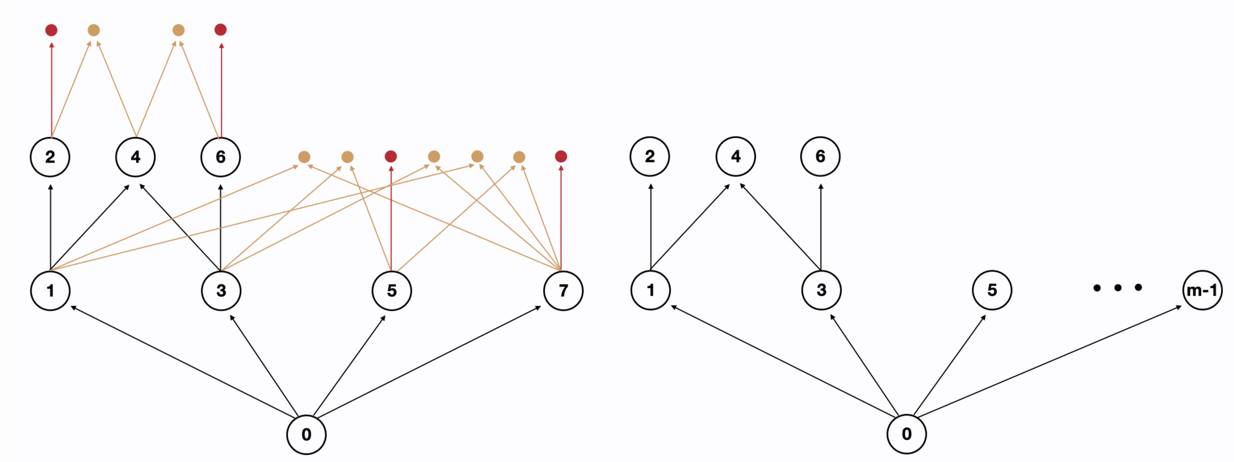

Figure 3: Outer Betti Elements in Kunz Poset of $S = \langle 8, 9, 11, 21, 15\rangle$ of the $S = \langle m, m+1, m+3, 2m+5, m+7, \ldots, m+(m-1)\rangle$ class of nmgs with mp size $|\rho| = \binom{m-3}{2} + 1$

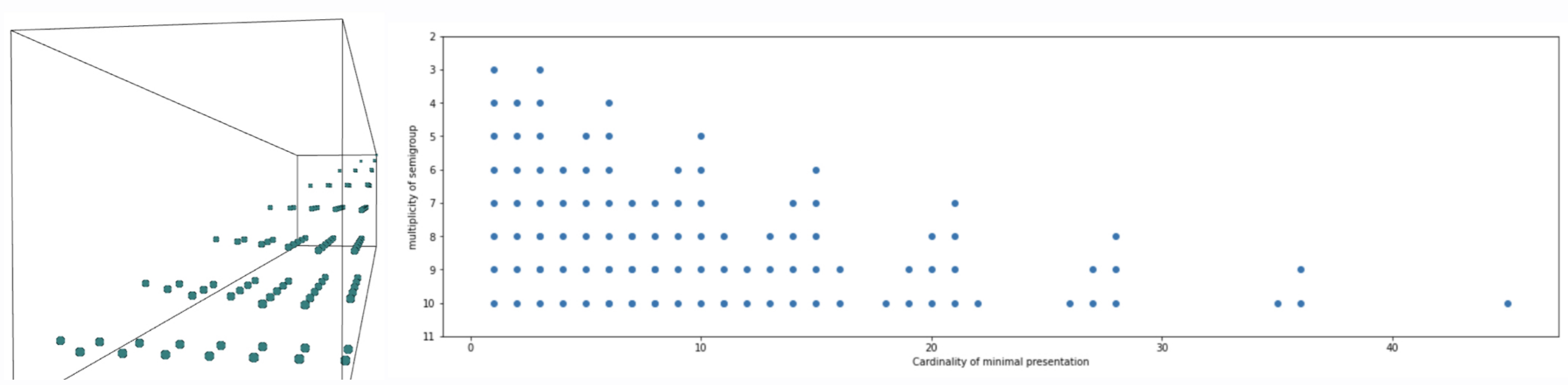

Figure 4: Minimal Presentation Sizes of Numerical Semigroups (QR code for website with the 3D plot)

Results

Theorem. (EKO) There are no numerical semigroups of multiplicity m with minimal presentation size between $\binom{m-1}{2}$ and $\binom{m}{2}$.

Theorem. (EKO) Let S be a numerical semigroup with embedding dimension $m-r$. Then the minimal presentation size of S is at least $\binom{m-r}{2} - r$.

Theorem. (EKO) The minimal presentation size of a numerical semigroups is given by the following formula

$$|\rho| = \binom{m-r}{2} + \sum_{1 \le i \le r} \left(d(p_i) - b(p_i) + x(p_i)\right)$$

where $d(p_i)$ is the number of relations occurring at p_1, $x(p_i)$ is the new outer Betti elements added in each step of the construction of P, and $b(p_i)$ is the number of outer Betti elements occuring at p_i.

Open Questions

Conjecture. (EKO) For $r < \frac{m}{2}$, the lower bound $\binom{m-r}{2} - r$ is an optimal lower bound for the minimal presentation size.

Question. What is the optimal lower bound for minimal presentation size when $r \geq \frac{m}{2}$?

Conjecture. (EKO) For all multiplicities, there exists a numerical semigroups of embedding dimension $m-5$ and minimal presentation size $\binom{m-5}{2} + 2$.